Monitoring Compositional Data using Multivariate EWMA

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Compositional Data (CoDa)

Definition

$\mathbf{x} = (x_1, x_2, \ldots, x_p)$ is defined as a p-part composition when (i) all its components are strictly positive real numbers (i.e. $\mathbf{x} \in \mathbb{R}_+^p$) and (ii) they carry only relative information.

Example of Muesli ($p = 3$)

66% of whole-grain cereals (barley / oat / wheat flakes), 24% of dried fruits (raisin, papaya, banana), 10% of nuts (almond, hazelnut, coconut).

Compositional equivalence

- $\mathbf{x} = (0.2, 0.5, 0.3)$,$\mathbf{y} = (20, 50, 30) \Rightarrow \mathbf{x} \neq \mathbf{y}$.
- But they are compositionally equivalent!
- Use of the closure function (in order to standardize):

$$\mathcal{C}(\mathbf{x}) = \left(\frac{\kappa x_1}{\sum_{i=1}^p x_i}, \frac{\kappa x_2}{\sum_{i=1}^p x_i}, \ldots, \frac{\kappa x_p}{\sum_{i=1}^p x_i} \right),$$

  $\kappa > 0$ is a constant to be fixed (usually $\kappa = 1$).
- Now, we have $\mathcal{C}(\mathbf{x}) = \mathcal{C}(\mathbf{y})$.

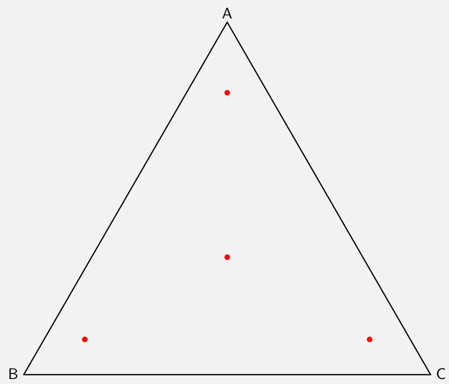
Compositional Data

Sample space

The sample space of compositional data is the simplex $\mathcal{S}^p$ defined as

$$\mathcal{S}^p = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_p) | x_i > 0, i = 1, 2, \ldots, p \text{ and } \sum_{i=1}^{p} x_i = \kappa. \right\}$$

$p = 3 \Rightarrow$ Ternary diagram



Examples

- $(1, 1, 1)$
- $(8, 1, 1)$
- $(1, 8, 1)$
- $(1, 1, 8)$

Aitchison's geometry

Euclidean geometry in $\mathbb{R}^p$

In $\mathbb{R}^p$ we have an Euclidean geometry that defines a vector space with a metric structure:

- how to add vectors,
- how to multiply them by a scalar value,
- how to know if two vectors are orthogonal,
- how to compute the distance between two points, etc.

Cannot be used in $\mathcal{S}^p$

Examples: $\mathbf{x} = (0.2, 0.5, 0.3) \in \mathcal{S}^p$, $\mathbf{y} = (0.2, 0.7, 0.1) \in \mathcal{S}^p$

- $\mathbf{x} + \mathbf{y} = (0.4, 1.2, 0.4) \notin \mathcal{S}^p$,
- $3 \times \mathbf{x} = (0.6, 1.5, 0.9) \notin \mathcal{S}^p$.

A new geometry in $\mathcal{S}^p$

With dedicated vector space and metric structure
$\Rightarrow$ the John Aitchison's geometry and also V. Pawlowsky-Glahn (Univ. of Girona) and J.J. Egozcue (Technical Univ. of Catalonia).

Aitchison's geometry

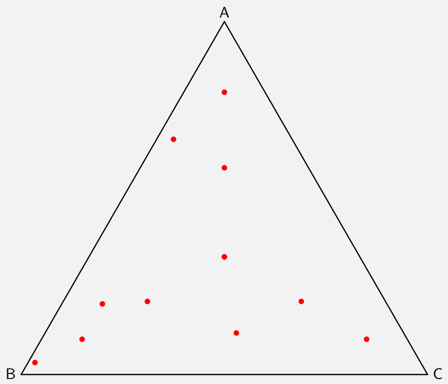
Perturbation operator $\oplus$ (analog to the translation in $\mathbb{R}^p$)

If $\mathbf{x} \in \mathcal{S}^p$ and $\mathbf{y} \in \mathcal{S}^p \Rightarrow \mathbf{x} \oplus \mathbf{y} = \mathbf{y} \oplus \mathbf{x} = \mathcal{C}(x_1 y_1, x_2 y_2, \ldots, x_p y_p)$.

Powering operator $\odot$ (analog to the scalar multiplication in $\mathbb{R}^p$)

If $a \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{S}^p \Rightarrow a \odot \mathbf{x} = \mathbf{x} \odot a = \mathcal{C}(x_1^a, x_2^a, \ldots, x_p^a)$.

Ternary diagram



Examples

- $\mathbf{x} \oplus (0.2, 0.7, 0.1)$
- $\mathbf{x} \odot 0.5$

Aitchison's geometry

$(\mathcal{S}^p, \oplus)$ is a commutative group structure

If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in $\mathcal{S}^p$

- $\mathbf{x} \oplus \mathbf{y} = \mathbf{y} \oplus \mathbf{x}$ (commutativity),
- $(\mathbf{x} \oplus \mathbf{y}) \oplus \mathbf{z} = \mathbf{x} \oplus (\mathbf{y} \oplus \mathbf{z})$ (associativity),
- $\mathbf{0}_{\mathcal{S}^p} = \mathcal{C}(1, 1, \ldots, 1) = (\frac{1}{p}, \frac{1}{p}, \ldots, \frac{1}{p})$ (neutral element for $\oplus$),
- $-\mathbf{x} = \mathcal{C}(x_1^{-1}, x_2^{-1}, \ldots, x_p^{-1})$ (inverse) such that $\mathbf{x} \oplus (-\mathbf{x}) = \mathbf{0}_{\mathcal{S}^p}$,
- $\mathbf{x} \ominus \mathbf{y} = \mathbf{x} \oplus (-\mathbf{y})$ (perturbation difference).

$\odot$ satisfies the properties of an external product

If $\mathbf{x}$, $\mathbf{y}$ in $\mathcal{S}^p$ and a, b in $\mathbb{R}$

- $a \odot (b \odot \mathbf{x}) = (ab) \odot \mathbf{x}$ (associativity),
- $a \odot (\mathbf{x} \oplus \mathbf{y}) = (a \odot \mathbf{x}) \oplus (a \odot \mathbf{y})$ and $(a + b) \odot \mathbf{x} = (a \odot \mathbf{x}) \oplus (b \odot \mathbf{x})$ (distributivity),
- $1 \odot \mathbf{x} = \mathbf{x} \odot 1 = \mathbf{x}$ (neutral element for $\odot$).

Aitchison's geometry

Centered logratio

$$\text{clr}(\mathbf{x}) = \left(\ln \frac{x_1}{\bar{x}_G}, \ln \frac{x_2}{\bar{x}_G}, \dots, \ln \frac{x_p}{\bar{x}_G} \right),$$

- $\bar{x}_G$ is the componentwise geometric mean of $\mathbf{x}$

$$\bar{x}_G = \left(\prod_{i=1}^{p} x_i \right)^{\frac{1}{p}} = \exp \left(\frac{1}{p} \sum_{i=1}^{p} \ln x_i \right).$$

- If $\text{clr}(\mathbf{x}) = \boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_p) \Rightarrow \xi_1 + \xi_2 + \dots + \xi_p = 0$.
- Remark: $\boldsymbol{\xi} \notin \mathcal{S}^p$.

Inverse centered logratio

If $\boldsymbol{\xi}$ satisfies $\xi_1 + \xi_2 + \dots + \xi_p = 0$ then

$$\text{clr}^{-1}(\boldsymbol{\xi}) = \mathcal{C}(\exp(\xi_1), \exp(\xi_2), \dots, \exp(\xi_p))$$

Aitchison's geometry

We already have $\oplus$, $\odot$. Now we need

Aitchison's inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \langle \text{clr}(\mathbf{x}), \text{clr}(\mathbf{y}) \rangle,$$

$\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^p$. Remark: $\mathbf{x}$ and $\mathbf{y}$ are compositionally orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle_A = 0$.

Aitchison's norm

$$||\mathbf{x}||_A = ||\text{clr}(\mathbf{x})||_2,$$

$|| \cdot ||_2$ is the L^2-norm in $\mathbb{R}^p$.

Aitchison's distance

$$d_A(\mathbf{x}, \mathbf{y}) = d_2(\text{clr}(\mathbf{x}), \text{clr}(\mathbf{y})),$$

$d_2(\cdot, \cdot)$ is the L^2-distance in $\mathbb{R}^p$.

Aitchison's geometry

Orthonormal decomposition

- Since $\mathbf{x}$ is constrained by $\sum_{i=1}^{p} x_i = \kappa \Rightarrow$ dimension of $\mathcal{S}^p$ is $p-1$.
- Let $\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_{p-1}\}$ be an orthonormal basis of $\mathcal{S}^p \Rightarrow$ $\langle \mathbf{b}_i, \mathbf{b}_j \rangle_A = \langle \text{clr}(\mathbf{b}_i), \text{clr}(\mathbf{b}_j) \rangle = 0\,(1)$ when $i \neq j$ $(i = j)$.
- Decomposition of $\mathbf{x} = (x_1^* \odot \mathbf{b}_1) \oplus (x_2^* \odot \mathbf{b}_2) \oplus \cdots \oplus (x_{p-1}^* \odot \mathbf{b}_{p-1})$ where $x_i^* = \langle \mathbf{x}, \mathbf{b}_i \rangle_A = \langle \text{clr}(\mathbf{x}), \text{clr}(\mathbf{b}_i) \rangle$.

Isometric logratio (and inverse)

$$\text{ilr}(\mathbf{x}) = \mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_{p-1}^*)$$

- If $\mathbf{B}$ is the $(p-1, p)$ matrix in which rows are $\text{clr}(\mathbf{b}_i)$, then

$$\text{ilr}(\mathbf{x}) = \text{clr}(\mathbf{x})\mathbf{B}^\intercal.$$

- The composition coordinates $\mathbf{x}$ can be obtained from the ilr-coordinates $\mathbf{x}^*$ using

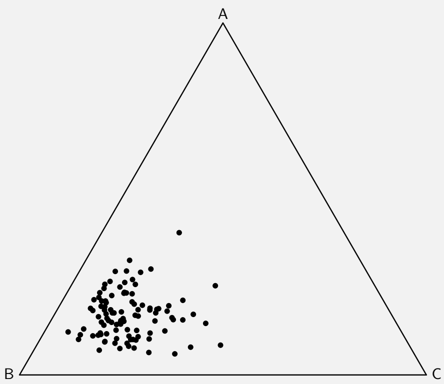
$$\text{ilr}^{-1}(\mathbf{x}^*) = \text{clr}^{-1}(\mathbf{x}^*\mathbf{B}) = \mathcal{C}(\exp(\mathbf{x}^*\mathbf{B})).$$

Aitchison's geometry

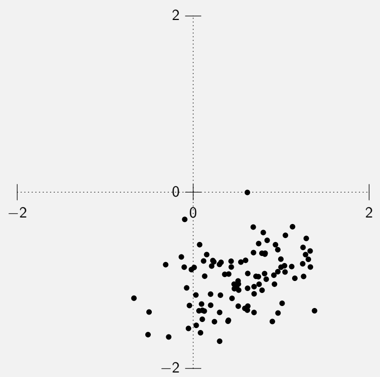
Matrix B

Many possible choices... A possible one is

$$B_{i,j} = \begin{cases} \sqrt{\frac{1}{(p-i)(p-i+1)}} & j \leq p-i \\ -\sqrt{\frac{p-i}{p-i+1}} & j = p-i+1 \\ 0 & j > p-i+1 \end{cases}$$



ilr
$\rightarrow$



MEWMA-CoDa chart

Assumptions

- At each sampling period $i = 1, 2, \ldots$, we collect n independent observations $\{\mathbf{X}_{i,1}, \ldots, \mathbf{X}_{i,n}\}$, where each $\mathbf{X}_{i,j} \in \mathcal{S}^p$, $j = 1, \ldots, n$.
- $\mathbf{X}^*_{i,j} = \text{ilr}(\mathbf{X}_{i,j}) \in \mathbb{R}^{p-1}$ are the corresponding ilr coordinates.
- $\mathbf{X}^*_{i,j} \sim \text{MNor}_{p-1}(\boldsymbol{\mu}^*, \boldsymbol{\Sigma}^*)$, where $\boldsymbol{\mu}^*$ is the $(1, p-1)$ mean vector, $\boldsymbol{\Sigma}^*$ is the $(p-1, p-1)$ variance-covariance matrix.
- When the process is in-control the composition center is $\boldsymbol{\mu}_0$ (or, equivalently $\boldsymbol{\mu}^*_0 = \text{ilr}(\boldsymbol{\mu}_0)$).
- When the process is out-of-control the composition center is $\boldsymbol{\mu}_1$ (or, equivalently $\boldsymbol{\mu}^*_1 = \text{ilr}(\boldsymbol{\mu}_1)$).

Goal is to monitor the composition center $\boldsymbol{\mu}$ using a Multivariate EWMA control chart monitoring $\boldsymbol{\mu}^*$.

MEWMA-CoDa chart

Monitored statistic

$$Q_i = \mathbf{Z}_i \boldsymbol{\Sigma}_Z^{-1} \mathbf{Z}_i^\intercal,\ i = 1, 2, \ldots$$

with $\mathbf{Z}_i \in \mathbb{R}^{p-1}$ defined as

$$\mathbf{Z}_i = r(\bar{\mathbf{X}}^*_i - \boldsymbol{\mu}^*_0) + (1-r)\mathbf{Z}_{i-1},\ i = 1, 2, \ldots$$

where $\mathbf{Z}_0 = \mathbf{0}$, $r \in (0, 1]$ is a smoothing parameter to be fixed and

$$\boldsymbol{\Sigma}_Z = \frac{r}{n(2-r)} \boldsymbol{\Sigma}^*.$$

Control limits

- Out-of-control signal when $Q_i > \text{UCL} = H$, where $H > 0$ is chosen to achieve a specified in-control ARL (difficult to compute).
- Non-centrality parameter $\delta = \sqrt{n(\boldsymbol{\mu}^*_1 - \boldsymbol{\mu}^*_0)^\intercal (\boldsymbol{\Sigma}^*)^{-1} (\boldsymbol{\mu}^*_1 - \boldsymbol{\mu}^*_0)}$.

MEWMA-CoDa chart – Comparison with the T^2_C chart

δ	$p = 3$	$p = 5$	$p = 10$	$p = 20$
		$\mathrm{ARL}_0 = 200$		
0.25	(64.6, 171.0)	(75.8, 181.0)	(91.1, 188.5)	(106.3, 192.7)
0.50	(26.4, 115.5)	(31.7, 138.1)	(39.6, 159.1)	(49.3, 172.7)
0.75	(15.1, 70.4)	(18.1, 94.4)	(22.8, 122.7)	(28.7, 145.0)
1.00	(9.9, 41.9)	(11.9, 61.0)	(15.0, 88.7)	(18.9, 115.2)
1.25	(7.1, 25.3)	(8.5, 38.7)	(10.7, 61.5)	(13.5, 87.6)
1.50	(5.4, 15.8))	(6.4, 24.6)	(8.1, 41.7)	(10.2, 64.4)
1.75	(4.3, 10.2)	(5.1, 16.0)	(6.4, 28.1)	(8.0, 46.4)
2.00	(3.5, 6.9)	(4.1, 10.6)	(5.2, 19.0)	(6.5, 33.0)
		$\mathrm{ARL}_0 = 500$		
0.25	(102.9, 416.4)	(126.1, 445.8)	(160.5, 467.4)	(197.5, 479.4)
0.50	(34.6, 265.7)	(41.8, 327.1)	(53.6, 385.5)	(69.3, 423.7)
0.75	(18.8, 151.9)	(22.6, 212.3)	(28.5, 287.3)	(36.0, 348.0)
1.00	(12.1, 84.9)	(14.4, 129.7)	(18.2, 199.1)	(23.0, 268.6)
1.25	(8.5, 48.3)	(10.1, 77.7)	(12.7, 131.8)	(16.1, 197.3)
1.50	(6.4, 28.3)	(7.6, 46.8)	(9.5, 85.1)	(11.9, 139.8)
1.75	(5.0, 17.3)	(5.9, 28.7)	(7.4, 54.5)	(9.3, 96.6)
2.00	(4.1, 11.0)	(4.8, 18.1)	(5.9, 35.1)	(7.4, 65.8)
		$\mathrm{ARL}_0 = 1000$		
0.25	(145.2, 816.8)	(184.1, 881.5)	(245.7, 929.2)	(315.8, 955.4)
0.50	(41.6, 499.9)	(50.7, 628.5)	(66.4, 753.4)	(88.9, 835.8)
0.75	(21.8, 272.9)	(26.0, 393.0)	(32.7, 547.6)	(41.6, 675.7)
1.00	(13.8, 145.9)	(16.3, 230.7)	(20.5, 368.1)	(26.0, 510.6)
1.25	(9.6, 79.4)	(11.3, 132.7)	(14.2, 235.5)	(17.9, 366.0)
1.50	(7.2, 44.6)	(8.4, 76.7)	(10.5, 146.8)	(13.2, 252.1)
1.75	(5.6, 26.1)	(6.5, 45.1)	(8.1, 90.7)	(10.2, 169.1)
2.00	(4.5, 15.9)	(5.3, 27.3)	(6.5, 56.3)	(8.1, 111.7)

Incorporating measurement errors

In practice

- Sample reduces to $n = 1$.
- Each $\mathbf{X}_i \in \mathcal{S}^p$ cannot be observed!
- But can be measured m times $\rightarrow \mathbf{Y}_{i,1}, \dots, \mathbf{Y}_{i,m}$.

Linearly covariate measurement error model

$$\mathbf{Y}_{i,k} = \mathbf{a} \oplus (b \odot \mathbf{X}_i) \oplus \varepsilon_{i,k},$$

- $\mathbf{a} \in \mathcal{S}^p$ and $b \in \mathbb{R}$ are perturbation and powering constants,
- $\varepsilon_{i,k} \sim \mathrm{MNOR}_{\mathcal{S}^p}(\mathbf{0}, \boldsymbol{\Sigma}^*_{\mathrm{M}})$ is an error term independent of $\mathbf{X}_i$.
- $\boldsymbol{\Sigma}^*_{\mathrm{M}}$ is the known measurement error variance-covariance matrix.

Incorporating measurement errors

Reduce the effect of measurement errors

Use

$$\begin{aligned}\bar{\mathbf{Y}}_i &= \frac{1}{m} \odot (\mathbf{Y}_{i,1} \oplus \cdots \oplus \mathbf{Y}_{i,m}) \\ &= \mathbf{a} \oplus (b \odot \mathbf{X}_i) \oplus \left(\frac{1}{m} \odot (\varepsilon_{i,1} \oplus \cdots \oplus \varepsilon_{i,m}) \right).\end{aligned}$$

We have $\bar{\mathbf{Y}}_i \sim \mathrm{MNOR}_{\mathcal{S}^p}(\boldsymbol{\mu}^*_{\bar{\mathbf{Y}}}, \boldsymbol{\Sigma}^*_{\bar{\mathbf{Y}}})$ with

$$\begin{aligned}\boldsymbol{\mu}^*_{\bar{\mathbf{Y}}} &= \mathbf{a}^* + b\boldsymbol{\mu}^*, \\ \boldsymbol{\Sigma}^*_{\bar{\mathbf{Y}}} &= b^2\boldsymbol{\Sigma}^* + \frac{1}{m}\boldsymbol{\Sigma}^*_{\mathrm{M}}.\end{aligned}$$

MEWMA-CoDa chart with measurement errors

Monitored statistic

$$Q_i = \mathbf{Z}^*_i \boldsymbol{\Sigma}^{-1}_{\mathbf{Z}^*_i} \mathbf{Z}^{*\intercal}_i$$

with $\mathbf{Z}^*_i \in \mathbb{R}^{p-1}$ defined as

$$\mathbf{Z}^*_i = r(\bar{\mathbf{Y}}^*_i - \mathbf{a}^* - b\boldsymbol{\mu}^*_0) + (1-r)\mathbf{Z}^*_{i-1}$$

where $\mathbf{Z}_0 = \mathbf{0}$, $r \in (0,1]$ is a smoothing parameter to be fixed and

$$\boldsymbol{\Sigma}_{\mathbf{Z}^*_i} = \frac{r}{(2-r)}\boldsymbol{\Sigma}_{\bar{\mathbf{Y}}^*} = \frac{r}{(2-r)}\left(b^2\boldsymbol{\Sigma}^* + \frac{1}{m}\boldsymbol{\Sigma}^*_{\mathrm{M}} \right).$$

MEWMA-CoDa chart with measurement errors

Non-centrality parameter

$$\delta_{\mathrm{M}} = b^2(\boldsymbol{\mu}_1^* - \boldsymbol{\mu}_0^*)\left(b^2\boldsymbol{\Sigma}^* + \frac{1}{m}\boldsymbol{\Sigma}_{\mathrm{M}}^*\right)^{-1}(\boldsymbol{\mu}_1^* - \boldsymbol{\mu}_0^*)^\intercal.$$

- Linna et al. (2001) "multivariate control charts are not equally powerful in detecting shifts in all directions in the presence of measurement errors".
- For a fixed δ, $\delta_{\mathrm{M}} \in [\delta_{\mathsf{min}}, \delta_{\mathsf{max}}]$ where $\delta_{\mathsf{min}} = \delta\lambda_1$ and $\delta_{\mathsf{max}} = \delta\lambda_{p-1}$.
- λ_1 (λ_{p-1}) is the smallest (largest) eigenvalues of the $(p-1, p-1)$ matrix $b^2\boldsymbol{\Sigma}^*(b^2\boldsymbol{\Sigma}^* + \frac{1}{m}\boldsymbol{\Sigma}_{\mathrm{M}}^*)^{-1}$.

Evaluation of the MEWMA-CoDa chart with measurement errors: influence of $\boldsymbol{\sigma}_{\mathbf{M}}$, b and m ... Not presented here (boring).

Illustrative example

The product

Muesli (for breakfast), where every 100 grams contain:

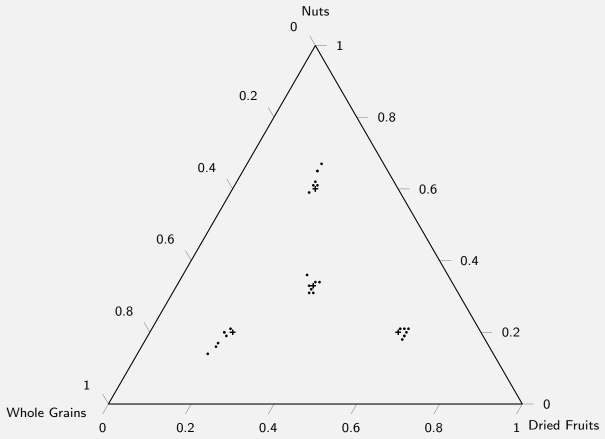
- A 66% of whole-grain cereals (barley flakes, oat flakes, wheat flakes),
- B 24% of dried fruits (raisin, papaya, banana),
- C 10% of nuts (almond, hazelnut, coconut).

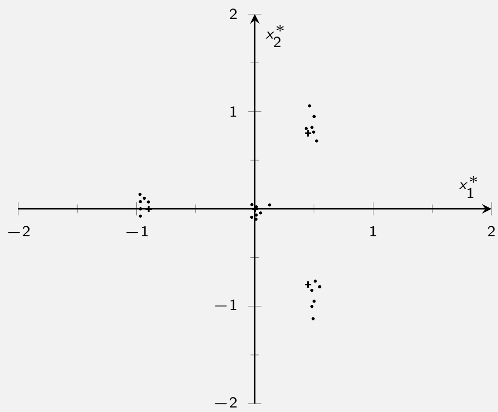
Phase 0 (calibration)

- $k = 4$ reference samples have been prepared with known percentages,
- measured $m = 7$ times.

i	j	$\mathbf{y}_i$			$\mathbf{x}_{i,j}$			$\mathbf{y}_i^*$		$\mathbf{x}_{i,j}^*$	
1	1	0.33	0.33	0.33	0.34	0.33	0.33	0.0000	0.0000	0.0122	0.0211
	2				0.32	0.35	0.33			0.0115	-0.0634
	3				0.33	0.35	0.32			0.0491	-0.0416
	4				0.31	0.35	0.34			-0.0259	-0.0858
	5				0.36	0.34	0.30			0.1255	0.0404
	6				0.34	0.32	0.34			-0.0247	0.0429
	7				0.31	0.36	0.33			0.0100	-0.1057
2	1	0.60	0.20	0.20	0.62	0.19	0.19	0.4485	0.7768	0.4828	0.8363
	2				0.65	0.17	0.18			0.5009	0.9484
	3				0.59	0.22	0.19			0.5224	0.6976
	4				0.61	0.20	0.19			0.4971	0.7885
	5				0.67	0.15	0.18			0.4621	1.0583
	6				0.65	0.17	0.18			0.5009	0.9484
	7				0.61	0.19	0.20			0.4343	0.8248
3	1	0.20	0.60	0.20	0.17	0.65	0.18	0.4485	-0.7768	0.5009	-0.9484
	2				0.14	0.69	0.17			0.4926	-1.1279
	3				0.21	0.60	0.19			0.5103	-0.7423
	4				0.19	0.62	0.19			0.4828	-0.8363
	5				0.20	0.62	0.18			0.5479	-0.8000
	6				0.16	0.66	0.18			0.4823	-1.0020
	7				0.20	0.62	0.18			0.5479	-0.8000
4	1	0.20	0.20	0.60	0.18	0.20	0.62	-0.8970	0.0000	-0.9668	-0.0745
	2				0.19	0.19	0.62			-0.9657	0.0000
	3				0.21	0.19	0.60			-0.8980	0.0708
	4				0.21	0.18	0.61			-0.9336	0.1090
	5				0.20	0.18	0.62			-0.9668	0.0745
	6				0.19	0.19	0.62			-0.9657	0.0000
	7				0.21	0.17	0.62			-0.9702	0.1494

Illustrative example





Estimation of the measurement parameters

- $\hat{\mathbf{a}} = (0.3354, 0.3357, 0.3289)$ or $\hat{\mathbf{a}}^* = (0.0162972, -0.0006318)$,
- $\hat{b} = 1.1070$,
- $\hat{\mathbf{\Sigma}}^*_{\mathrm{M}} = \begin{pmatrix} 0.0014346 & 0.0007812 \\ 0.0007812 & 0.0102893 \end{pmatrix}$.

Illustrative example

Phase 1 (in-control parameters estimation)

20 batches of muesli have been measured $m = 3$ times.

$$\hat{\boldsymbol{\mu}}^*_{\tilde{\mathbf{Y}}} = (1.2766, 0.7657) \qquad \hat{\mathbf{\Sigma}}^*_{\tilde{\mathbf{Y}}} = \begin{pmatrix} 0.0146362 & 0.0105839 \\ 0.0105839 & 0.0510887 \end{pmatrix}$$

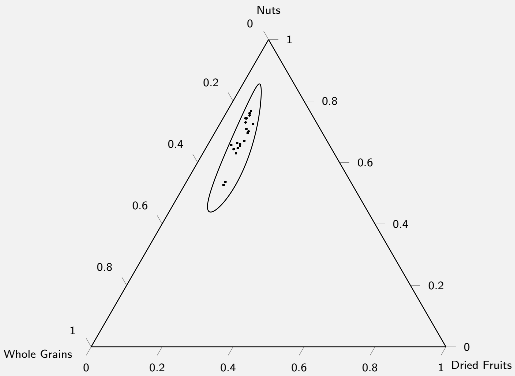
We deduce

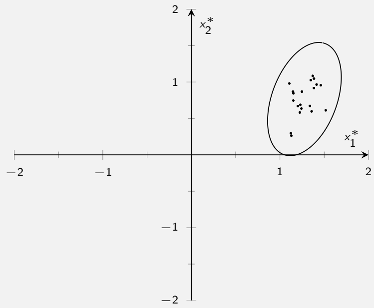
- $\hat{\boldsymbol{\mu}}^*_0 = \frac{1}{\hat{b}}(\hat{\boldsymbol{\mu}}^*_{\tilde{\mathbf{Y}}} - \hat{\mathbf{a}}^*) = (1.1385, 0.6922)$,
- $\hat{\mathbf{\Sigma}}^* = \frac{1}{\hat{b}^2}\left(\hat{\mathbf{\Sigma}}^*_{\tilde{\mathbf{Y}}} - \frac{1}{m}\hat{\mathbf{\Sigma}}^*_{\mathrm{M}}\right) = \begin{pmatrix} 0.0115533 & 0.0084242 \\ 0.0084242 & 0.038891 \end{pmatrix}$.

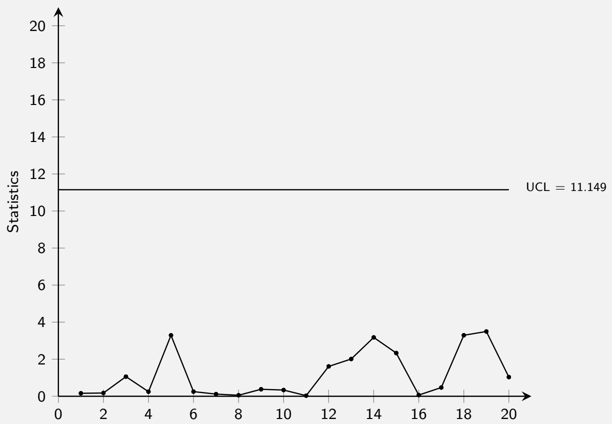
Upper control limit

The optimal couple for $\delta = 1.5$ is $(r = 0.226, \mathrm{UCL} = H = 11.149)$.

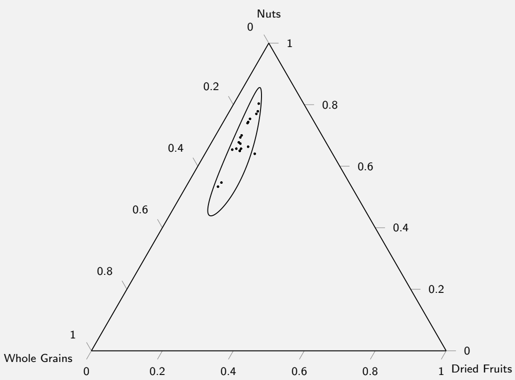
Illustrative example

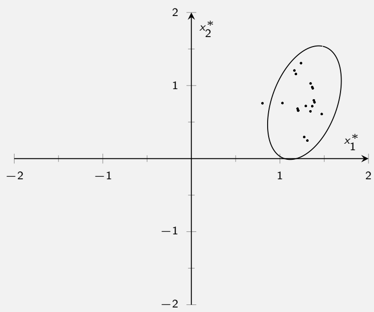


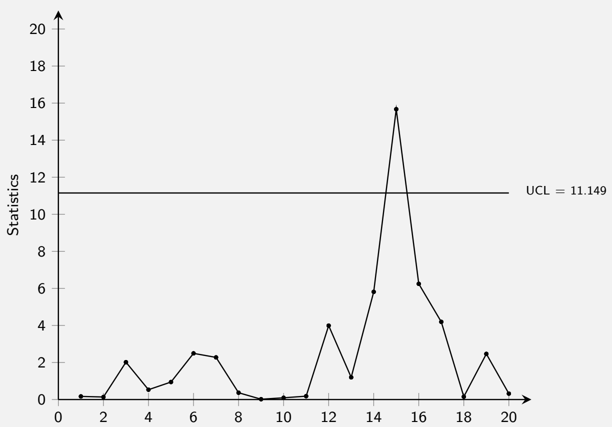




Illustrative example







THANK YOU FOR YOUR ATTENTION

K.P. Tran, P. Castagliola, G. Celano, M.B.C Khoo. Monitoring Compositional Data using Multivariate Exponentially Weighted Moving Average Scheme. Quality and Reliability Engineering International, 34(3):391-402, 2018.